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Entering

AP Calculus BC

Summer Math Packet

AP Calculus BC Summer Packet Part 1

Review of Algebra, Geometry, and Pre-Calculus

Summer Enrichment 2021

Congratulations cadet. You are part of a very select few taking the hardest and most rewarding math class at FCS. It is the hardest because BC Calculus covers a full year of college-level calculus. It is the most rewarding because if you do well on the AP Exam you get two semesters of college credit! This class will not be easy and will require you to put in a lot of effort both in class and outside of class beginning now, summer 2021. But I promise you it will all be worth it when you are able to exempt Calculus 1 and Calculus 2 in college! And on top of the credit, you will be learning a subject that is extremely fascinating and gratifying to complete ☺

Young cadet, your mission to make a 5 begins right now, Summer 2021, with your completion of summer packets part 1 and part 2. Part 1 covers Algebra, Geometry, and Pre-Calculus topics. Part 2 covers topics from AP Calculus AB. Beginning now I will be treating you like college students: the answers to every problem in the packets are included. You need to have the maturity and responsibility to complete each problem and not just copy the answers. If you cut corners it will catch up to you eventually. For packet 1 you need to complete the problems and show your work on a separate piece of notebook paper. For packet 2 you can solve and put your answers on the packet itself. Bring both completed packets on the first day of class.

Cadet, your mission to making a 5 begins now. I look forward to seeing you at mission control in August.

-Mr. Lyerly

Formulas and Identities

Trigonometric Identities:

Reciprocal Identities:

$$sinx = \frac{1}{cscx} \qquad cosx = \frac{1}{secx} \qquad tanx = \frac{1}{cotx}$$

$$cscx = \frac{1}{sinx} \qquad secx = \frac{1}{cosx} \qquad cotx = \frac{1}{tanx}$$

Quotient Identities:

$$tanx = \frac{sinx}{cosx} \qquad cotx = \frac{cosx}{sinx}$$

Pythagorean Identities:

$$sin^2x + cos^2x = 1 \qquad 1 + tan^2x = sec^2x \qquad 1 + cot^2x = csc^2x$$

Geometric Formulas:

Area of a Trapezoid: $A = \frac{1}{2}h(b_1 + b_2)$

Area of a Triangle: $A = \frac{1}{2}bh$

Area of an Equilateral Triangle: $A = \frac{\sqrt{3}}{4}s^2$

Area of a Circle: $A = \pi r^2$

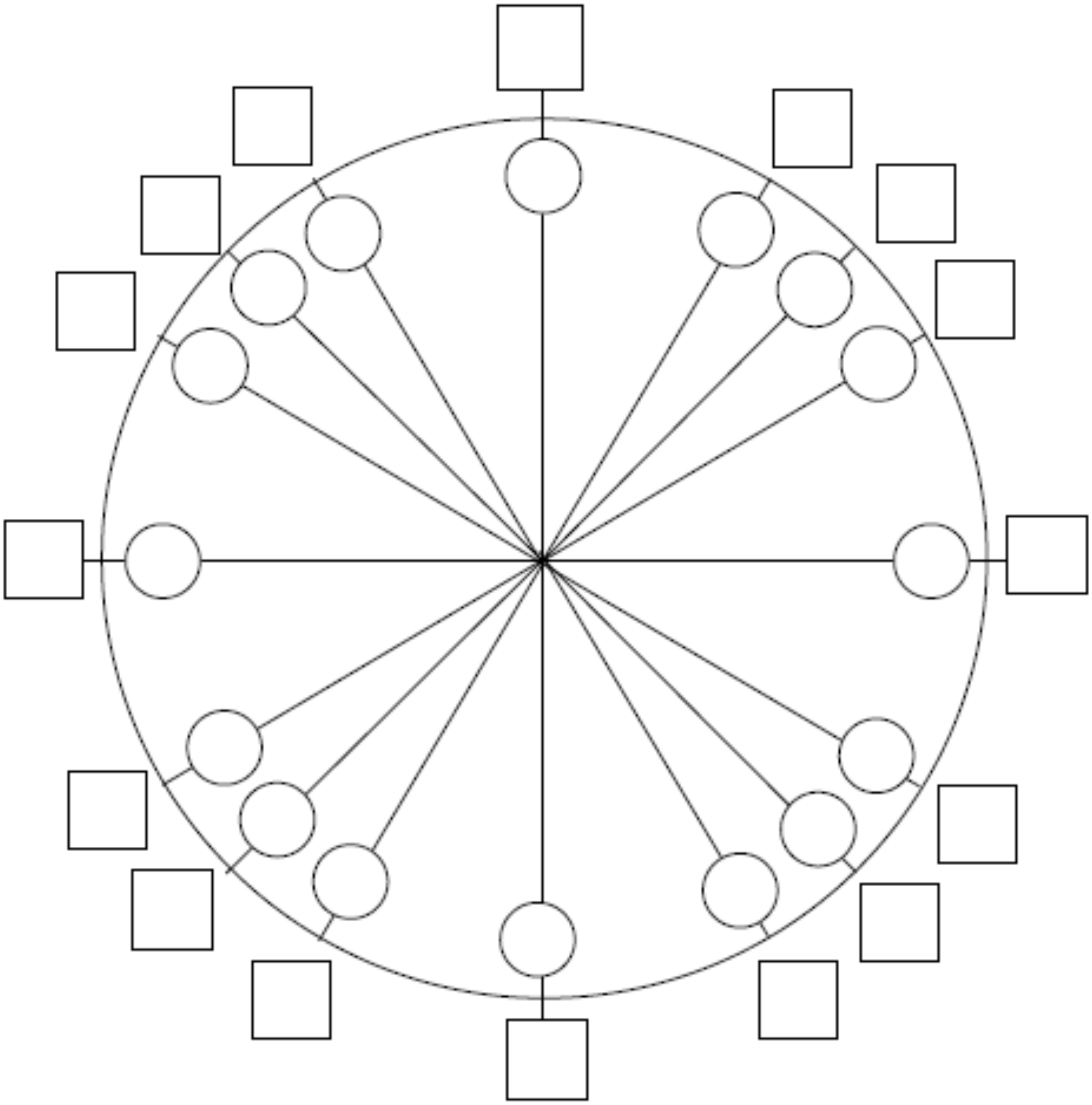
Circumference of a Circle: $C = 2\pi r$ or $C = d\pi$

Volume of a Cylinder: $V = \pi r^2 h$

Volume of a Sphere: $V = \frac{4}{3}\pi r^3$

Volume of a Right Circular Cone: $V = \frac{1}{3}\pi r^2 h$

UNIT CIRCLE



Place **degree** measures in the **circles**

Place **radian** measures in the **squares**

Place $(\boldsymbol{cos\theta}, \boldsymbol{sin\theta})$ in parenthesis **outside the square**

Place $\boldsymbol{tan\theta}$ **outside the parenthesis**

SKILLS NEEDED FOR CALCULUS

I. Algebra:

A. Exponents* *(Operations with integer, fractional and negative exponents)*

B. Factoring* *(GCF, trinomials, difference of squares and cubes, sum of cubes, grouping)*

C. Rationalizing* *(numerator and denominator)*

D. Solving Algebraic Equations and Inequalities* *(linear, quadratic, rational, radical, and absolute value)*

II. Graphing and Functions

A. Lines* *(intercepts, slopes, write equations using point-slope and slope intercept, parallel, perpendicular, distance and midpoint formulas)*

B. Functions* *(definition, notation, domain, range, inverse, composition)*

C. Basic Shape and Transformations* *(absolute value, rational, root, higher order curves, logarithms, natural log, exponential, trigonometric, piece-wise, and inverse functions)*

III. Geometry

A. Pythagorean Theorem

B. Area Formulas *(circles, polygons, surface area of solids)*

C. Volume Formulas

D. Similar Triangles

IV. Logarithmic and Exponential Functions

A. Simplify Expressions* *(using laws of logarithms and exponents)*

B. Solve Logarithmic and Exponential Equations* *(include ln as well as log)*

C. Sketch Graphs*

D. Inverses*

V. Trigonometry

A. Unit Circle* *(definition of functions, angles in radians and degrees)*

B. Use Pythagorean Identities and Formulas to Simplify Expressions and Prove Identities

C. Solve Equations*

D. Inverse Trigonometric Functions*

E. Right Triangle Trigonometry

F. Graphs*

* A solid working foundation in these areas is critical

AP Calculus BC Packet 1 Problems

Work the following problems on your own paper. Show all necessary work.
Number each problem and stay organized throughout.

I. Algebra

A. Exponents:

Simplify the expression; your answer should only contain positive exponents.

1) $\dfrac{(8x^3yz)^{\frac{1}{3}}(2x)^3}{4x^{\frac{1}{3}}\left(yz^{\frac{2}{3}}\right)^{-1}}$

B. Factor Completely:

2) $9x^2 + 3x - 3xy - y$ (hint: use grouping)

3) $64x^6 - 1$

4) $42x^4 + 35x^2 - 28$

5) $15x^{\frac{5}{2}} - 2x^{\frac{3}{2}} - 24x^{\frac{1}{2}}$ (hint: factor a GCF of $x^{\frac{1}{2}}$ fist)

6) $x^{-1} - 3x^{-2} + 2x^{-3}$ (hint: factor GCF of x^{-3} first)

C. Rationalizing Denominator / Numerators:

7) $\frac{3-x}{1-\sqrt{x-2}}$

8) $\frac{\sqrt{x+1}+1}{x}$

D. Simplify the rational expression:

9) $\frac{(x+1)^3(x-2)+3(x+1)^2}{(x+1)^4}$

E. Solve: *You may use your graphing calculator to check your solutions.*

10) $(x-3)^2 > 4$

11) $\frac{x+5}{x-3} \leq 0$

12) $3x^3 - 14x^2 - 5x \leq 0$

13) $x < \frac{1}{x}$

14) $\frac{x^2-9}{x+1} \geq 0$

15) $\frac{1}{x-1} + \frac{4}{x-6} > 0$

16) $x^2 < 4$

17) $|2x+1| < \frac{1}{4}$

F. Solve the System. *Solve the system algebraically and then check the solution by graphing each function and using your calculator to find points of intersection.*

18) $x - y + 1 = 0$

$y - x^2 = -5$

19) $x^2 - 4x + 3 = y$

$-x^2 + 6x - 9 = y$

II. Graphing and Functions

A. Linear graphs:

20) Passes through the point $(2, -1)$ and has the slope $-\frac{1}{3}$

21) Passes through the point $(4, -3)$ and is perpendicular to $3x + 2y = 4$

22) Passes through the point $(-1, -2)$ and is parallel to $y = \frac{3}{5}x - 1$

B. Functions: *Find the domain of the following.*

23) $f(x) = \frac{3}{x-2}$ 24) $g(x) = \log(x-3)$

25) $h(x) = \sqrt{2x-3}$ 26) $w(x) = \frac{\sqrt{x-1}}{x^2-1}$

27) Given f(x) below, sketch the graph over the domain $[-3,3]$.

$$s(x) = \begin{cases} x & \text{if } x \geq 0 \\ 1 & \text{if } -1 \leq x < 0 \\ x-2 & \text{if } x < -1 \end{cases}$$

Find the composition/inverses as indicated below.

Let $f(x) = x^2 + 3x - 2$ $g(x) = 4x - 3$ $h(x) = \ln x$ $w(x) = \sqrt{x-4}$

28) $g^{-1}(x)$ 29) $h^{-1}(x)$ 30) $w^{-1}(x)$, for $x \geq 4$ 31) $f(g(x))$ 32) $h\Big(g\big(f(1)\big)\Big)$

33) Does $y = 3x^2 - 9$ have an inverse function? Explain your answer.

Let $f(x) = 2x$ $g(x) = -x$ $h(x) = 4$

34) $(f \circ g)(x)$ 35) $(f \circ g \circ h)(x)$

36) Let $s(x) = \sqrt{4-x}$ and $t(x) = x^2$, find the domain and range of $(s \circ t)(x)$.

C. Basic Shapes of Curves:

Sketch the graphs. You may use your graphing calculator to verify your graph, but you should be able to graph the following by knowledge of the shape of the curve, by plotting a few points, and by your knowledge of transformations.

37) $y = \sqrt{x}$ 38) $y = \ln x$ 39) $y = \frac{1}{x}$ 40) $y = |x-2|$ 41) $y = \frac{1}{x-2}$ 42) $\frac{x}{x^2-4}$

43) $y = e^{-x}$ 44) $f(x) = \begin{cases} \sqrt{25-x^2} & \text{if } x < 0 \\ \dfrac{x^2-25}{x-5} & \text{if } x \geq 0, x \neq 5 \\ 0 & \text{if } x = 5 \end{cases}$

III. Logarithmic and Exponential Functions

A. Simplifying Expressions: **Non-calculator**

45) $log_4\left(\frac{1}{16}\right)$ 46) $3log_3 3-\frac{3}{4}log_3 81+\frac{1}{3}log_3\left(\frac{1}{27}\right)$ 47) $log_9 27$

48) $log_{125}\left(\frac{1}{5}\right)$ 49) $log_w w^{45}$ 50) $\ln e$ 51) $\ln 1$ 52) $\ln e^2$

B. Solve Equations: **Non-calculator**

53) $log_6(x+3)+log_6(x+4)=1$ 54) $\log x^2-\log 100=\log 1$ 55) $3^{x+1}=15$

IV. Trigonometry:

A. Unit Circle: *Know the unit circle – radian and degree measure. Be prepared for a quiz.*

56) State the domain, range and fundamental period of each function.

a) $y=\sin x$ b) $y=\cos x$ c) $y=\tan x$

B. Solve the Equations

57) $cos^2x=cosx+2;\ \ 0\le x\le 2\pi$ 58) $2\sin(2x)=\sqrt{3};\ \ 0\le x\le 2\pi$

59) $cos^2x+sinx+1=0;\ \ 0\le x\le \pi$

C. Inverse Trig Functions: note: $Sin^{-1}x=\text{Arcsin}\,x$

60) $\text{Arcsin}\,1$ 61) $\text{Arcsin}\left(-\frac{\sqrt{2}}{2}\right)$ 62) $\text{Arccos}\left(\frac{\sqrt{3}}{2}\right)$ 63) $\sin\left(Arccos\left(\frac{\sqrt{3}}{2}\right)\right)$

64) State the domain and range for each: a) $\text{Arcsin}\,(x)$ b) $Arccos(x)$ c) $Arctan(x)$

D. Be able to do the following on your graphing calculator:

Be familiar with the calculator commands to find values, roots, minimums, maximums, and intersections. You may need to zoom in on areas of your graph to find the information.

Answers should be accurate to 3 decimal places. Sketch each graph.

65-68. Given the following function $f(x)=2x^4-11x^3-x^2+30x$.

65) Find all roots. 66) Find all local maxima. 67) Find all local minima.

> Local maxima or local minima are the points on the graph where there is a highest or lowest point within an interval, such as a vertex on a parabola.

68) Find the following: $f(-1),\ f(2),\ f(0),\ f(.125)$

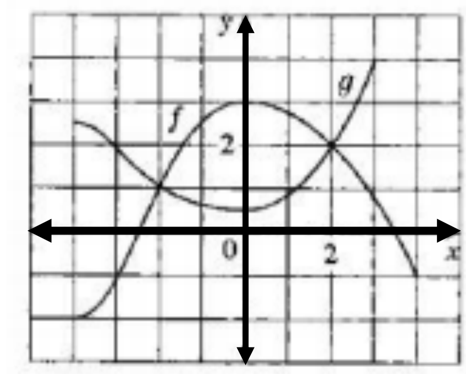
69) Graph the following two functions and find their points of intersection using the intersect command on your calculator.

$y=x^3+5x^2-7x+2$ and $y=.2x^2+10$ Window: x min: -10 x max: 10 scale 1

Y min: -10 y max: 50 scale 0

V. Functions and Models

70) The graphs of f and g are given.



a) State the values of f(-4) and $g(3)$

b) For what values of x if $f(x) = g(x)$?

c) Estimate the solution of the equation $f(x) = 1$.

d) On what interval is f decreasing?

e) State the domain and range for f.

f) State the domain and range for g.

71) If $f(x) = 3x^2 - x + 2$, find $f(2),\ f(a),\ f(-a),\ f(a+1),\ 2f(a),\ f(a^2),\ [f(a)]^2$.

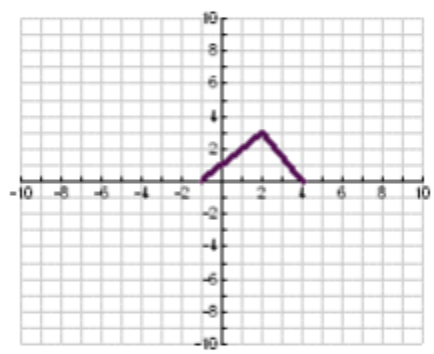
72) Find the domain of each function.

a) $f(x) = \frac{x}{3x-1}$

b) $g(u) = \sqrt{u} + \sqrt{4-u}$

73) Find the expression for the bottom half of the parabola $x + (y-1)^2 = 0$.

74) Find the expression for the function whose graph is the given curve. (hint: piece-wise function)

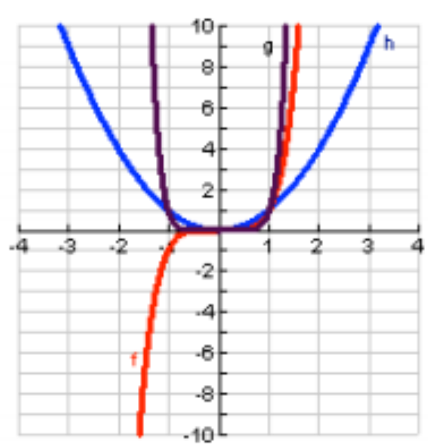


75) Match each equation with its graph. Explain your choices. (Don't use a computer or graphing calculator).

a) $y = x^2$

b) $y = x^5$

c) $y = x^8$

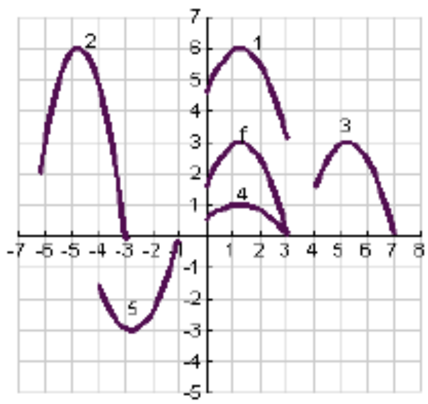


76) Suppose the graph of $y = f(x)$ is given. Write the equations for the graphs that are obtained from the graph of f (hint: use your knowledge of transformations).:

a) shift 3 units upward
b) shift 3 units downward
c) shift 3 units to the right
d) shift 3 units to the left
e) reflect about the x-axis
f) reflect about he y-axis
g) stretch vertically by a factor of 3
h) shrink vertically by a factor of 3

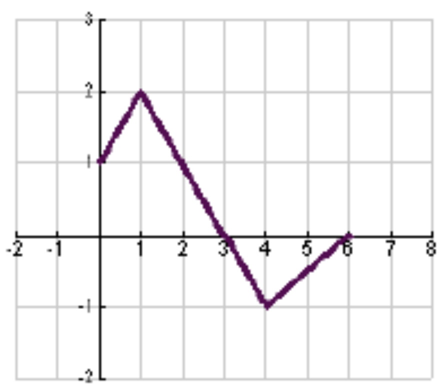
77) The graph $y = f(x)$ is given. Math each equation with its graph and given reasons for your choices.

a) $y = f(x-4)$ b) $y = f(x) + 3$ c) $y = \frac{1}{3}f(x)$ d) $y = -f(x+4)$ e) $y = 2f(x+6)$



78) The graph of $y = f(x)$ is given. Use it to graph the following functions.

a) $f(2x)$ b) $f\left(\frac{1}{2}x\right)$ c) $y = f(-x)$ d)$y = -f(-x)$



79) Find the functions $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$ as well as their domains.

$f(x) = \sin x$ $g(x) = 1 - \sqrt{x}$

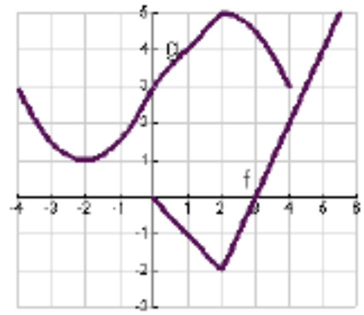
80) Express the function in the form $f \circ g$

$F(x) = (x^2 + 1)^{10}$

81) Use the given graphs of f and g to evaluate each expression, or explain why it is undefined.

a) $f(g(2))$ b) $g(f(0))$ c) $(f \circ g)(0)$

d) $(g \circ f)(6)$ e) $(g \circ g)(-2)$ f) $(f \circ f)(4)$



82) Graph the ellipse $4x^2 + 2y^2 = 1$ by graphing the functions whose graphs the upper and lower halves of the ellipse.

83) Use your calculator to find all solutions of the equation accurate to three decimal places

$$x^3 - 9x^2 - 4 = 0$$

84) Starting with the graph of $y = e^x$, write the equation of the graph that results from

a) shifting 2 units downward
b) shifting 2 units to the right
c) reflecting about the x-axis
d) reflecting about the y-axis
e) reflecting about the x-axis and then about the y-axis

For #85-87, find the formula for the inverse of the function.

85) $f(x) = \sqrt{10 - 3x}$ 86) $f(x) = e^{x^3}$ 87) $f(x) = \ln(x + 3)$

For #88-89, find the exact value of each expression (non-calculator)

88) a) $log_2 64$ b) $log_6 \frac{1}{36}$

89) b) $log 1.25 + log 80$ b) $log_5 10 + log_5 20 - 3 log_5 2$

90) Express the given quantity as a single logarithm.

$$2 \ln 4 - \ln 2$$

Part 1 Answers:

1. $4x^{11/3}y^{4/3}z$ 2. $(3x + 1)(3x - y)$ 3. $(2x - 1)(4x^2 + 2x + 1)(2x + 1)(4x^2 - 2x + 1)$

4. $7(3x^2 + 4)(2x^2 - 1)$ 5. $x^{1/2}(3x - 4)(5x + 6)$

6. $x^{-3}(x - 2)(x - 1)$ 7. $1 + \sqrt{x - 2}$ 8. $\dfrac{1}{\sqrt{x+1} - 1}$ 9. $\dfrac{x^2 - x + 1}{(x + 1)^2}$

10. $x > -1$ or $x < -5$ 11. $-5 \le x < 3$ 12. $x \le -\frac{1}{3}$ or $0 \le x \le 5$ 13. $0 < x < 1$ or $x < -1$

14. $[-3, -1) \cup [3, \infty)$ 15. $x > 6$ or $1 < x < 2$

16. $-2 < x < 2$ 17. $-\frac{5}{8} < x < -\frac{3}{8}$ 18. (3, 4), (-2, -1) 19. (2, -1), (3, 0)

20. $y = -\frac{1}{3}x - \frac{1}{3}$ 21. $y = \frac{2}{3}x - \frac{17}{3}$ 22. $y = \frac{3}{5}x - \frac{7}{5}$ 23. D: $x \neq 2$ 24. $D: x > 3$

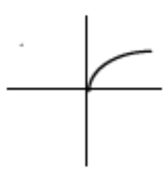
25. D: $x \geq 3/2$ 26. $x > -1$ and $x \neq 1$ 27. R: $-5 \le y < -3$ or $0 \le y \le 3$ 28. $g^{-1}(x) = \dfrac{x+3}{4}$

29. $h^{-1}(x) = e^x$ 30. $y = x^2 + 4 \quad x \geq 0$, 31. $f(g(x)) = 16x^2 - 12x - 2$ 32. ln 5

33. no, explain: this function Is not one-to-one (pass the horizontal line test)

34. $-2x$ 35. - 8 36. D: $-2 \le x \le 2$ R: $0 \le y \le 2$

37.



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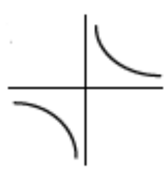

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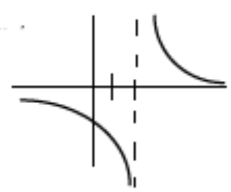

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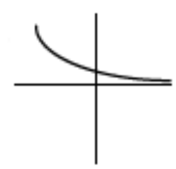

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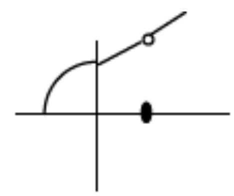


45. - 2 46. –1 47. 3/2 48. – 1/3 49. 45 50. 1 51. 0 52. 2 53. - 1

54. x = 10, -10 55. $\dfrac{\log 15}{\log 3} - 1$ 56. a) D: all reals, R: $-1 \leq x \leq 1$, 2π

b) D: all reals, R: $-1 \leq x \leq 1$, 2π c) D: $x \neq \pi/2$ R: all reals, π

57. π 58. $\frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$ 59. $\frac{3\pi}{2}$ 60. $\frac{\pi}{2}$

61. $-\frac{\pi}{4}$ 62. $\frac{\pi}{6}$ 63. $\frac{1}{2}$ 64. Arcsin(x) D: [-1, 1] Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

Arc cos (x) D: [-1, 1]. R: $[0, \pi]$ Arctan (x) D: all reals R: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

65. - 1.5, 0 , 2 , 5 66. rel max. (1.07, 20.1) 67. rel. min (- .89, - 18. 48), (3.94, - 88)

68. f(-1) = - 18 f(2)=0 f(0)=0 f(0.125)=3.7133 69. 3 points of intersection - one is (-5.77, 16.66)

69. Do not use trace to find points. Use CALC commands.

70. (a) -2,4 (b) -2 and 2 (c) x= -3 and 4 (d) [0,4] (e) d: [-4,4] r:[-2,3] (f) d: [-4,3] r:[0.5,4]

71. f(2)=12, f(-2) = 16, f(a) = 2a^2 -a +2,
f(-a)=3a^2+a+2, f(a+1)=3a^2+5a+4, 2f(a)=6a^2-2a+4, f(2a)=12a^2-2a+2, f(a^2)=3a^4 -a^2 +2,
[f(a)]^2=9a^4 -6a^3+13a^2-4a+4, f(a+h)=3a^2 +6ah+3h^2 -a-h+2

72. (a) $(-\infty, 1/3) \cup (1/3, \infty)$ (b) [0,4] 73. $f(x) = 1 - \sqrt{-x}$ (domain: $x \le 0$)

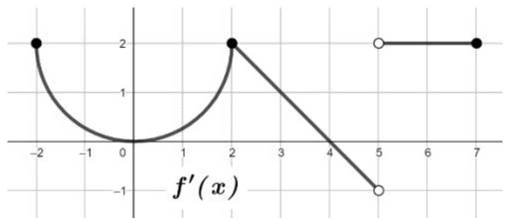
74. $f(x) = \begin{cases} x+1 & if\ -1 \le x \le 2 \\ -\frac{3}{2}x+6 & if\ \ 2 < x \le 4 \end{cases}$

75. (a) matches with h (b) matches with f and (c) matches with g.
76. (a) y = f(x)+3 (b) y = f(x) -3 (c) y = f(x-3) (d) y = f(x+3) (e) y = -f(x) (f) y = f(-x) (g) y = 3f(x) (h) y = 1/3 f(x) 113. (a) graph 3 (b) graph 1 (c)graph 4 (d) graph 5 (e) graph 2 77. (a) shrink horizontally by a factor of 2 (b) stretch horizontally by a factor of 2 (c) reflect the graph of f about the y-axis (d) reflect the graph of f about the y-axis, then about the x-axis

79. (f+g)(x) =x^3+5x^2-1 d: all real numbers (f-g)(x)= x^3-x^2+1 d: all reals (fg)(x)=3x^5+6x^4-x^3-2s^2 d: all reals (f/g)(x)=(x^3 +2x^2)/(3x^2 -1) d: x cannot Equal $\pm\frac{1}{\sqrt{3}}$ 79. $(f \circ g)(x) = \sin(1-\sqrt{x}) \quad d:[0,\infty)$ $(g \circ f)(x) = 1-\sqrt{\sin x} \quad d:[0,\pi],[2\pi,3\pi]\ etc$

$(f \circ f)(x) = \sin(\sin x) \quad d:(-\infty,\infty)$ $(g \circ g)(x) = 1-\sqrt{1-\sqrt{x}} \quad d:[0,1]$ 80. g(x)=x^2 +1 and F(x)=x^10 81. (a) 4 (b) 3 (c)0 (d) not defined (e) 4 (f) -2 82. graph y = +sqr((1-4x^2)/2) and -sqr((1-4x^2)/2) 83. about 9.05 84. (a) y = e^x -2 (b) y = e^(x-2) (c) y = -e^x (d) y = e^(-x) (e) y = - e^(-x) 85. $f^{-1}(x) = -\frac{1}{3}x^2 + \frac{10}{3}$ d: [0,∞) 86. $f^{-1}(x) = \sqrt[3]{\ln x}$ 87. $f^{-1}(x) = e^x - 3$
88 (a) 6 (b) -2 89. (a) 2 (b) 2 90. ln8

AP Calculus BC Summer Packet Part 2
A Journey Through Calculus AB from A to Z



$$h(x) = \begin{cases} \dfrac{\sin(x-1)}{x-1}, & x < 1 \\ kx^2 - 8x + 6, & 1 \le x \le 3 \\ 4e^{2x-6} - x^2 + 5, & x > 3 \end{cases}$$

x	$g''(x)$
-4	13
-1	10
0	8
2	e
3	-1

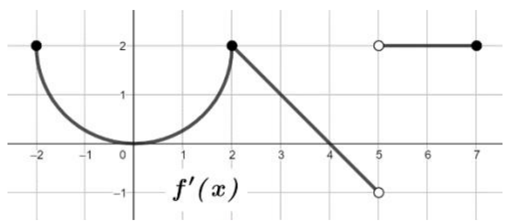
Consider $f'(x)$, the derivative of the continuous function f, defined on the closed interval $[-6, 7]$ except at $x = 5$. A portion of f' is given in the graph above and consists of a semi-circle and two line segments. The function $h(x)$ is a piecewise defined function given where k is a constant. The function $g(x)$ and its derivatives are differentiable. Selected values for the decreasing function $g''(x)$, the second derivative of g are given in the table above.

(**A**) Find the value of k such that $h(x)$ is continuous at $x = 3$. Show your work.

(**B**) Using the value of k found in part (A), is $h(x)$ continuous at $x = 1$? Justify your answer.

(**C**) Is there a time $c, -4 < c < 3$ such that $g'''(c) = -2$? Give a reason for your answer.

(**D**) For each $x = 2$ and $x = 4$, determine if $f(x)$ has a local minimum, local maximum or neither. Give a reason for your answer.



$$h(x) = \begin{cases} \dfrac{\sin(x-1)}{x-1}, & x < 1 \\ kx^2 - 8x + 6, & 1 \le x \le 3 \\ 4e^{2x-6} - x^2 + 5, & x > 3 \end{cases}$$

x	$g''(x)$
-4	13
-1	10
0	8
2	e
3	-1

(**E**) Find all x value(s) on the open interval $(-2, 5)$ where $f(x)$ has a point of inflection. Give a reason for your answer.

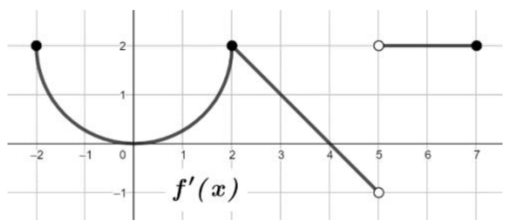
(**F**) Find the average rate of change of $h(x)$, in terms of k, over the interval $[2,5]$.

(**G**) If $f(3) = 5$, write an equation of the tangent line to $f(x)$ at $x = 3$.

(**H**) Use a right Riemann sum with the four subintervals indicated in the table to approximate $\int_{-4}^{3} g''(x)\,dx$.

Is this approximation an over or under estimate? Give a reason for your answer.

(**I**) Evaluate $\int_{0}^{7} f'(x)\,dx$.



$$h(x) = \begin{cases} \dfrac{\sin(x-1)}{x-1}, & x < 1 \\ kx^2 - 8x + 6, & 1 \le x \le 3 \\ 4e^{2x-6} - x^2 + 5, & x > 3 \end{cases}$$

x	$g''(x)$
-4	13
-1	10
0	8
2	e
3	-1

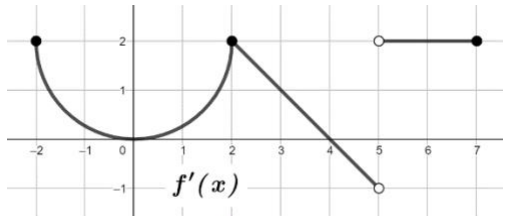
(**J**) Let $k(x) = x^2 + \int_1^x f'(t)dt$. Find the values for $k'(2)$ and $k''(2)$ or state that it does not exist.

(**K**) Find $h'(4)$.

(**L**) Let $m(x) = f'(x)g'\left(\frac{x}{2}\right)$. Find $m'(6)$.

(**M**) Let $p(x) = f(x^2 - 1)$. Find $p'(2)$.

(**N**) Find the average value of $f'(x)$ over the interval [2,5].



$$h(x) = \begin{cases} \dfrac{\sin(x-1)}{x-1}, & x < 1 \\ kx^2 - 8x + 6, & 1 \le x \le 3 \\ 4e^{2x-6} - x^2 + 5, & x > 3 \end{cases}$$

x	$g''(x)$
-4	13
-1	10
0	8
2	e
3	-1

(**O**) Evaluate $\displaystyle\int_{-1}^{3} [2g'''(x) + 7]dx$

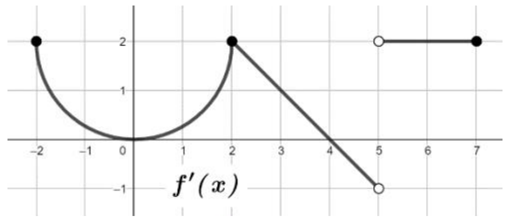
(**P**) If $\displaystyle\int_{-6}^{2} f'(x)dx = 5 - 2\pi$, then find $\displaystyle\int_{-2}^{-6} f'(x)dx$.

(**Q**) For $0 \le t \le 2.5$, a particle is moving along a horizontal axis with velocity $v(t) = \ln\big(g''(t)\big)$.
Is the particle speeding up or slowing down at time $t = 2$? Give a reason for your answer.

(**R**) Let x be the number of people, in thousands, inside an amusement park. The number of people inside the park that have contracted a virus can be modeled by $v(x) = \dfrac{h(x)}{3x}$ for $3 < x < 5$.

The number of people in the park is increasing at a constant rate of 0.2 thousands of people per minute.

Using this model, what is the rate that people inside the park are contracting the virus with respect to time when there are four thousand people in the park?



$$h(x) = \begin{cases} \dfrac{\sin(x-1)}{x-1}, & x < 1 \\ kx^2 - 8x + 6, & 1 \le x \le 3 \\ 4e^{2x-6} - x^2 + 5, & x > 3 \end{cases}$$

x	$g''(x)$
-4	13
-1	10
0	8
2	e
3	-1

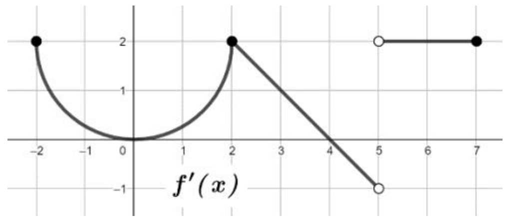
(**S**) $\lim\limits_{x\to 2} \dfrac{\int_4^x f'(t)dt + x}{\sin(x^2-4)}$

(**T**) Let $k = 0$, evaluate $\displaystyle\int_2^4 h(x)dx$.

(**U**) Is there a time $c, -4 < c < 3$, such that $g''(c) = 0$? Give a reason for your answer.

(**V**) Estimate $g'''(-2)$. Show the calculations that lead to your answer.

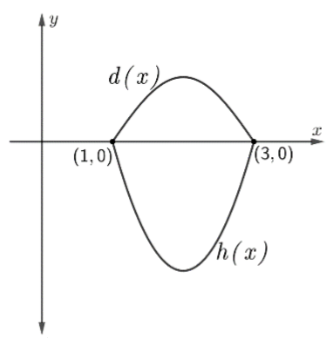
(**W**) For $-6 \le x \le -2, f'(x) = \dfrac{1}{4}(x+4)^3$. If $f(-2) = 0$, find the minimum value of $f(x)$ on $[-6,2]$.



$$h(x) = \begin{cases} \dfrac{\sin(x-1)}{x-1}, & x < 1 \\ kx^2 - 8x + 6, & 1 \le x \le 3 \\ 4e^{2x-6} - x^2 + 5, & x > 3 \end{cases}$$

x	$g''(x)$
-4	13
-1	10
0	8
2	e
3	-1

(**X**) Let $y = r(x)$ be the particular solution to the differential equation $\dfrac{dy}{dx} = \dfrac{h(x) + x^2}{y}$ for $x > 3$.
Find the particular solution $y = r(x)$ given the initial condition $(4, -2e)$.



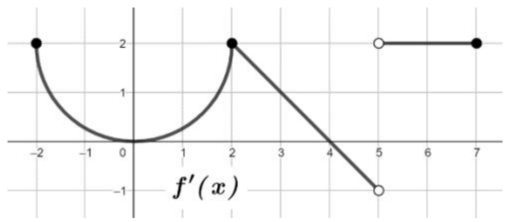
(**Y**) The graphs of $d(x) = -\sin\left(\dfrac{\pi x}{2} + \dfrac{1}{2}\right)$ and $h(x)$ are shown above for $1 \le x \le 3$ when $k = 2$.
Find the area bounded by the graphs of $d(x)$ and $h(x)$.

(**Z**) Set up, but do not evaluate, an expression involving one or more integrals that gives the volume when the region bounded by the graphs above is revolved about the line $y = -5$.

Special thanks to my friends Bryan Passwater, Speedway HS, Speedway, Indiana for authoring this activity and to Vernon "Ted" Gott, retired teacher from Harwood, MD for providing detailed solutions.

Solutions Summer Packet Part 2

A Journey Through Calculus AB from A to Z



$$h(x)=\begin{cases}\dfrac{\sin(x-1)}{x-1}, & x<1\\ kx^2-8x+6, & 1\le x\le 3\\ 4e^{2x-6}-x^2+5, & x>3\end{cases}$$

x	$g''(x)$
-4	13
-1	10
0	8
2	e
3	-1

Consider $f'(x)$, the derivative of the continuous function f, defined on the closed interval $[-6,7]$ except at $x=5$. A portion of f' is given in the graph above and consists of a semi-circle and two line segments. The function $h(x)$ is a piecewise defined function given where k is a constant. The function $g(x)$ and its derivatives are differentiable. Selected values for the decreasing function $g''(x)$, the second derivative of g are given in the table above.

(**A**) Find the value of k such that $h(x)$ is continuous at $x=3$. Show your work.

continuous at $x=3 \Rightarrow \lim\limits_{x\to3^-}h(x)=\lim\limits_{x\to3^+}h(x)=h(3)$

$\lim\limits_{x\to3^-}h(x)=\lim\limits_{x\to3^-}k(3)^2-8(3)+6=9k-18=h(3)$ $\qquad$ $\lim\limits_{x\to3^+}h(x)=4e^0-(3)^2+5=0$

$9k-18=0\Rightarrow \boxed{k=2}$

(**B**) Using the value of k found in part (A), is $h(x)$ continuous at $x=1$? Justify your answer.

$\lim\limits_{x\to1^-}h(x)=\lim\limits_{x\to1^-}\dfrac{\sin(x-1)}{x-1}\Rightarrow$ indeterminant form $\dfrac{0}{0}\Rightarrow \lim\limits_{x\to1^-}\dfrac{\sin(x-1)}{x-1}=\underbrace{\lim\limits_{x\to1^-}\dfrac{\cos(x-1)}{1}}_{\text{l'Hospital's Rule}}=\dfrac{1}{1}=1$

$\lim\limits_{x\to1^+}h(x)=\lim\limits_{x\to1^+}(2x^2-8x+6)=0\Rightarrow \lim\limits_{x\to1^-}h(x)\ne\lim\limits_{x\to1^+}h(x)\Rightarrow \boxed{\text{not continuous at } x=1 \text{ when } k=2}$

(**C**) Is there a time $c, -4<c<3$ such that $g'''(c)=-2$? Give a reason for your answer.

$\dfrac{g''(3)-g''(-4)}{3-(-4)}=\dfrac{-1-13}{3-(-4)}=-2$ Since $g''(x)$ is differentiable on the interval $-4<x<3$, the MVT guarantees

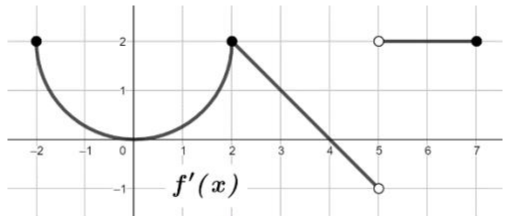
there is a $x=c, -4<c<3$, such that $g'''(c)=-2$ because the average rate of change $\dfrac{g''(3)-g''(-4)}{3-(-4)}=-2$ on the

interval $-4<x<3$.

(**D**) For each $x=2$ and $x=4$, determine if $f(x)$ has a local minimum, local maximum or neither. Give a reason for your answer.

$f(x)$ has neither at $x=2$ because $f'(x)$ does not change signs (positive $\leftrightarrow$ negative) at $x=2$.

$f(x)$ has a local maximum at $x=4$ because $f'(x)$ changes from positive to negative at $x=4$.



$$h(x) = \begin{cases} \dfrac{\sin(x-1)}{x-1}, & x < 1 \\ kx^2 - 8x + 6, & 1 \le x \le 3 \\ 4e^{2x-6} - x^2 + 5, & x > 3 \end{cases}$$

x	$g''(x)$
-4	13
-1	10
0	8
2	e
3	-1

(E) Find all x value(s) on the open interval $(-2, 5)$ where $f(x)$ has a point of inflection. Give a reason for your answer.

$f(x)$ has a point of inflection at $x = 0$ and $x = 2$

becasue $f'(x)$ changes from increasing to decreasing (or vice versa) at these x – values.

(F) Find the average rate of change of $h(x)$, in terms of k, over the interval $[2,5]$.

average rate of change of $h(x)$ on $[2,5] = \dfrac{h(5)-h(2)}{5-2} = \dfrac{\left(4e^{(2\cdot5-6)} - (5)^2 + 5\right) - \left(k(2)^2 - 8(2) + 6\right)}{3}$

$= \dfrac{(4e^4 - 20) - (4k - 10)}{3} = \dfrac{(4e^4 - 4k - 10)}{3}$

(G) If $f(3) = 5$, write an equation of the tangent line to $f(x)$ at $x = 3$.

tangent line: $y = f(3) + f'(3)(x-3) = 5 + (x-3)$

(H) Use a right Riemann sum with the four subintervals indicated in the table to approximate $\int_{-4}^{3} g''(x)\,dx$.

Is this approximation an over or under estimate? Give a reason for your answer.

$\int_{-4}^{3} g''(x)\,dx \approx (-1-(-4))(g''(-1)) + (0-(-1))(g''(0)) + (2-0)(g''(2)) + (3-2)(g''(3))$

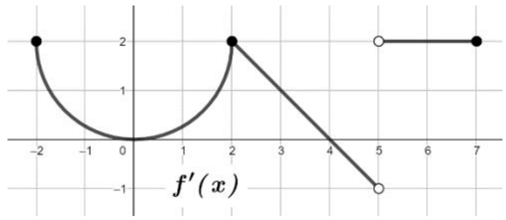
$= (3)(10) + (1)(8) + (2)(e) + (1)(-1) = 37 + 2e$

This is an underestimate of $\int_{-4}^{3} g''(x)\,dx$ because $g''(x)$ is given to be decreasing.

(I) Evaluate $\int_{0}^{7} f'(x)\,dx$.

$\int_{0}^{7} f'(x)\,dx = \int_{0}^{2} f'(x)\,dx + \int_{2}^{4} f'(x)\,dx + \int_{4}^{5} f'(x)\,dx + \int_{5}^{7} f'(x)\,dx$

$= \left((2)(2) - \frac{1}{4}\pi(2)^2\right) + \left(\frac{1}{2}(2)(2)\right) + \left(\left(\frac{1}{2}(-1)(1)\right)\right) + ((2)(2)) = (4-\pi) + \left(\frac{11}{2}\right) = \frac{19}{2} - \pi$



$$h(x)=\begin{cases}\dfrac{\sin(x-1)}{x-1}, & x<1\\ kx^2-8x+6, & 1\le x\le 3\\ 4e^{2x-6}-x^2+5, & x>3\end{cases}$$

x	$g''(x)$
-4	13
-1	10
0	8
2	e
3	-1

(**J**) Let $k(x)=x^2+\int_1^x f'(t)dt$. Find the values for $k'(2)$ and $k''(2)$ or state that it does not exist.

$k'(x)=2x+f'(x)\Rightarrow k'(2)=2(2)+f'(2)=4+2=6$

$k''(x)=2+f''(x)\Rightarrow k''(2)=2+f''(2)\Rightarrow$ does not exist because $f'(x)$ is not differentiable at $x=2$

(**K**) Find $h'(4)$.

$x>3, h'(x)=\dfrac{d}{dx}\left(4e^{2x-6}-x^2+5\right)=4e^{2x-6}(2)-2x\Rightarrow h'(4)=4e^{2(4)-6}(2)-2(4)=8e^2-8$

(**L**) Let $m(x)=f'(x)g'\left(\frac{x}{2}\right)$. Find $m'(6)$.

$$m'(x)=f''(x)g'\left(\frac{x}{2}\right)+f'(x)g''\left(\frac{x}{2}\right)\left(\frac{1}{2}\right)$$

$$m'(6)=f''(6)g'(3)+f'(6)g''(3)\left(\frac{1}{2}\right)=(0)g'(3)+(2)\left(\frac{1}{2}\right)(-1)=-1$$

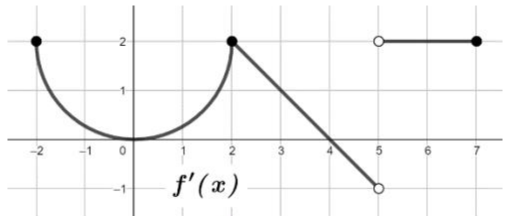
(**M**) Let $p(x)=f(x^2-1)$. Find $p'(2)$.

$p'(x)=f'\left(x^2-1\right)(2x)\Rightarrow p'(2)=f'\left((2)^2-1\right)(2(2))=f'(3)(4)=(1)(4)=4$

(**N**) Find the average value of $f'(x)$ over the interval $[2,5]$.

average value of $f'(x)$ on $[2,5]=\dfrac{1}{5-2}\displaystyle\int_2^5 f'(x)\,dx=\frac{1}{3}\int_2^5 f'(x)\,dx=\frac{1}{3}\left(\int_2^4 f'(x)\,dx+\int_4^5 f'(x)\,dx\right)$

$$=\frac{1}{3}\left(\left(\frac{1}{2}(2)(2)\right)+\left(\left(\frac{1}{2}(-1)(1)\right)\right)\right)=\frac{1}{3}\left(\frac{3}{2}\right)=\frac{1}{2}$$



$$h(x) = \begin{cases} \dfrac{\sin(x-1)}{x-1}, & x < 1 \\ kx^2 - 8x + 6, & 1 \le x \le 3 \\ 4e^{2x-6} - x^2 + 5, & x > 3 \end{cases}$$

x	$g''(x)$
-4	13
-1	10
0	8
2	e
3	-1

(**O**) Evaluate $\int_{-1}^{3} [2g'''(x) + 7]dx$

$$\int_{-1}^{3}(2g'''(x)+7)dx = \left[2g''(x)+7x\right]_{-1}^{3} = (2g''(3)+7(3)) - (2g''(-1)+7(-1))$$

$$(2(-1)+21) - (2(10)-7) = 19 - 13 = 6$$

(**P**) If $\int_{-6}^{2} f'(x)dx = 5 - 2\pi$, then find $\int_{-2}^{-6} f'(x)dx$.

$$\int_{-6}^{2} f'(x)\,dx = \int_{-6}^{-2} f'(x)\,dx + \int_{-2}^{2} f'(x)\,dx = \int_{-2}^{2} f'(x)\,dx - \int_{-2}^{-6} f'(x)\,dx$$

$$5 - 2\pi = \left[(4)(2) - \frac{1}{2}\pi(2)^2\right] - \int_{-2}^{-6} f'(x)\ \ dx$$

$$5 - 2\pi = [8 - 2\pi] - \int_{-2}^{-6} f'(x)\,dx$$

$$\int_{-2}^{-6} f'(x)\,dx = [8 - 2\pi] - [5 - 2\pi] = 3$$

(**Q**) For $0 \le t \le 2.5$, a particle is moving along a horizontal axis with velocity $v(t) = \ln(g''(t))$. Is the particle speeding up or slowing down at time $t = 2$? Give a reason for your answer.

$v(2) = \ln(g''(2)) = \ln(e) = 1 > 0$ $\qquad$ $v'(t) = \dfrac{1}{g''(t)} g'''(t)$

$v'(2) = \dfrac{1}{g''(2)} g'''(2) = \dfrac{1}{e}(g'''(2)) < 0$ because $g''(t)$ is decreasing

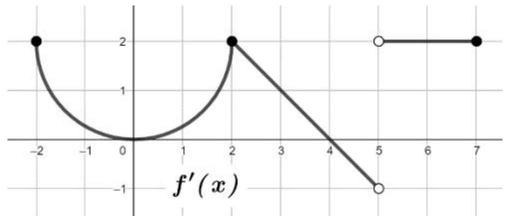
speed $= |v(t)|$ is decreasing because $v(2) > 0$ and $v'(2) < 0$, when positive numbers decrease, the absolute value decreases.

(**R**) Let x be the number of people, in thousands, inside an amusement park. The number of people inside the park that have contracted a virus can be modeled by $v(x) = \dfrac{h(x)}{3x}$ for $3 < x < 5$.

The number of people in the park is increasing at a constant rate of 0.2 thousands of people per minute. Using this model, what is the rate that people inside the park are contracting the virus with respect to time when there are four thousand people in the park?

$$x > 3 \Rightarrow v(x) = \frac{h(x)}{3x} = \frac{4e^{2x-6} - x^2 + 5}{3x} \Rightarrow v'(x) = \frac{(3x)(8e^{2x-6} - 2x) - 3(4e^{2x-6} - x^2 + 5)}{(3x)^2}$$

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} \Rightarrow \left.\frac{dv}{dt}\right|_{x=4} = (v'(4))(0.2) = \frac{(12)(8e^2 - 8) - 3(4e^2 - 11)}{(12)^2}(0.2) = \frac{(28e^2 - 21)}{(48)(5)} = 0.7745\ldots$$



$$h(x) = \begin{cases} \dfrac{\sin(x-1)}{x-1}, & x < 1 \\ kx^2 - 8x + 6, & 1 \le x \le 3 \\ 4e^{2x-6} - x^2 + 5, & x > 3 \end{cases}$$

x	$g''(x)$
-4	13
-1	10
0	8
2	e
3	-1

(S) $\lim\limits_{x\to 2} \dfrac{\int_4^x f'(t)dt + x}{\sin(x^2-4)}$

$$\lim_{x\to 2}\left(\int_4^x f'(t)\,dt + x\right) = \int_4^2 f'(t)\,dt + 2 = -\int_2^4 f'(t)\,dt + 2 = -\left(\frac{1}{2}(2)(2)\right) + 2 = 0 \qquad \lim_{x\to 2}\sin\left(x^2-4\right) = \sin(0) = 0$$

$$\lim_{x\to 2}\frac{\int_4^x f'(t)\,dt + x}{\sin\left(x^2-4\right)} = \underbrace{\lim_{x\to 2}\frac{f'(t)+1}{\cos\left(x^2-4\right)(2x)}}_{\text{l'Hospital's Rule}} = \frac{2+1}{\cos(0)(4)} = \frac{3}{4}$$

(T) Let $k = 0$, evaluate $\displaystyle\int_2^4 h(x)dx$.

$$\int_2^4 h(x)\,dx = \int_2^3 h(x)\,dx + \int_3^4 h(x)\,dx = \int_2^3 (-8x+6)\,dx + \int_3^4 \left(4e^{2x-6} - x^2 + 5\right)dx$$

$$= \left[-4x^2 + 6x\right]_2^3 + \left[2e^{2x-6} - \frac{1}{3}x^3 + 5x\right]_3^4 = [-14] + \left[\left(2e^2 - \frac{28}{3}\right)\right] = 2e^2 - \frac{70}{3}$$

(U) Is there a time $c, -4 < c < 3$, such that $g''(c) = 0$? Give a reason for your answer.

$g''(x)$ is differentiable $\Rightarrow$ $g''(x)$ is continuous

$g''(2) = e > 0$ and $g''(3) = -1 < 0$

Applying the IVT, there is time $c, 2 < c < 3$ such that $g''(c) = 0$

$2 < c < 3 \Rightarrow -4 < c < 3 \Rightarrow$ there is time $c, -4 < c < 3$ such that $g''(c) = 0$

(V) Estimate $g'''(-2)$. Show the calculations that lead to your answer.

$$g'''(-1) \approx \frac{g''(-1) - g''(-4)}{(-1)-(-4)} = \frac{(10)-(13)}{3} = -1$$

(W) For $-6 \le x \le -2, f'(x) = \dfrac{1}{4}(x+4)^3$. If $f(-2) = 0$, find the minimum value of $f(x)$ on $[-6,2]$.

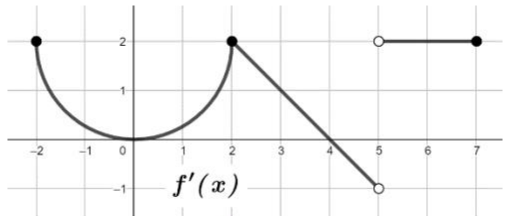
$-6 \le x \le -2 \Rightarrow f'(x) = \dfrac{1}{4}(x+4)^3 = 0 \Rightarrow x = -4 \qquad -2 < x \le 2 \Rightarrow f'(x) = 0 \Rightarrow x = 0$

$$x = -6 \Rightarrow f(-6) = f(-2) - \int_{-6}^{-2} f'(x)\,dx = 0 - \int_{-6}^{-2} f'(x)\,dx = -\left[\frac{1}{16}(x+4)^4\right]_{-6}^{-2} = -\left[1-(1)\right] = 0$$

$$x = -4 \Rightarrow f(-4) = -\int_{-4}^{-2} f'(x)\,dx = -\int_{-4}^{-2} f'(x)\,dx = -\left[\frac{1}{16}(x+4)^4\right]_{-4}^{-2} = -\left[1-(0)\right] = -1$$

$$x = 0 \Rightarrow f(0) = \int_{-2}^{0} f'(x)\,dx = 4 - \frac{1}{4}\pi(2)^2 = 4 - \pi \qquad x = 2 \Rightarrow f(2) = \int_{-2}^{2} f'(x)\,dx = 8 - \frac{1}{2}\pi(2)^2 = 8 - 2\pi$$

minimum value of $f(x)$ on $[-6,2]$ is -1 when $x = -4$



$$h(x)=\begin{cases}\dfrac{\sin(x-1)}{x-1}, & x<1\\ kx^2-8x+6, & 1\le x\le 3\\ 4e^{2x-6}-x^2+5, & x>3\end{cases}$$

x	$g''(x)$
-4	13
-1	10
0	8
2	e
3	-1

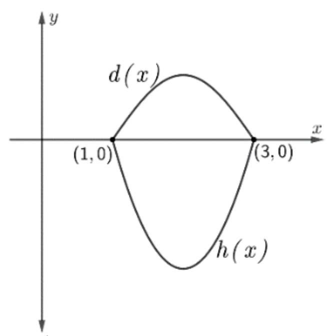
(**X**) Let $y=r(x)$ be the particular solution to the differential equation $\dfrac{dy}{dx}=\dfrac{h(x)+x^2}{y}$ for $x>3$. Find the particular solution $y=r(x)$ given the initial condition $(4,-2e)$.

$$y\,dy=\left(h(x)+x^2\right)dx$$

$$\int y\,dy=\int\left(\left(4e^{2x-6}-x^2+5\right)+x^2\right)dx\Rightarrow\frac{1}{2}y^2=2e^{2x-6}-\frac{1}{3}x^3+5x+\frac{1}{3}x^3+C$$

$$(4,-2e)\Rightarrow\frac{1}{2}(-2e)^2=2e^{2(4)-6}+5(4)+C\Rightarrow 2e^2=2e^2+20+C\Rightarrow C=-20$$

$$\frac{1}{2}y^2=2e^{2x-6}+5x-20\Rightarrow y^2=4e^{2x-6}+10x-40\Rightarrow y=-\sqrt{4e^{2x-6}+10x-40}=r(x)$$



(**Y**) The graphs of $d(x)=-\sin\left(\dfrac{\pi x}{2}+\dfrac{1}{2}\right)$ and $h(x)$ are shown above for $1\le x\le 3$ when $k=2$. Find the area bounded by the graphs of $d(x)$ and $h(x)$.

$$\int_1^3\left(d(x)-h(x)\right)dx=\left[\left(\frac{2}{\pi}\right)\cos\left(\frac{\pi x}{2}+\frac{1}{2}\right)\right]_1^3-\left[\frac{2}{3}x^3-4x^2+6x\right]_1^3 \quad \text{Interesting fact: } \cos\left(\frac{3\pi}{2}+C\right)=-\cos\left(\frac{\pi}{2}+C\right)$$

$$=\left[\left(\frac{2}{\pi}\right)\cos\left(\frac{3\pi}{2}+\frac{1}{2}\right)-\left(\frac{2}{\pi}\right)\cos\left(\frac{\pi}{2}+\frac{1}{2}\right)\right]-\left[\left(-\frac{8}{3}\right)\right]=\left(\frac{4}{\pi}\right)\cos\left(\frac{3\pi}{2}+\frac{1}{2}\right)+\frac{8}{3}$$

(**Z**) Set up, but do not evaluate, an expression involving one or more integrals that gives the volume when the region bounded by the graphs above is revolved about the line $y=-5$.

$$\pi\int_1^3\left(\left(-5-d(x)\right)^2-\left(-5-h(x)\right)^2\right)dx \text{ or } \pi\int_1^3\left(\left(d(x)-(-5)\right)^2-\left(h(x)-(-5)\right)^2\right)dx$$

Special thanks to my friends Bryan Passwater, Speedway HS, Speedway, Indiana for authoring this activity and to Vernon "Ted" Gott, retired teacher from Harwood, MD for providing detailed solutions.